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Modal-Space Control of Distributed Gyroscopic Systems

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This paper presents a control scheme for distributed gyroscopic systems based on modal synthesis. The control parameters for each controlled mode are chosen independently of those for any other mode, thus resulting in a diagonal gain matrix for the controlled modes. The modal control laws are implemented by means of a reduced-order observer that estimates linear combinations of the modal state vector. The proposed scheme considers uncontrolled modes simultaneously and has the advantage that it minimizes truncation effects and that it exhibits no observation spillover instability due to modeled uncontrolled modes. The paper also includes specification of positions of actual control forces and torques and sensors.

I. Introduction

THE development of the space shuttle has opened the possibility of constructing very large spacecraft. The spacecraft can be constructed in modular form, with various substructures manufactured on the ground and assembled in space, or with the substructures both manufactured and assembled in space. By necessity, large spacecraft must be extremely flexible, so that the mathematical simulation of such spacecraft must be in terms of high-order systems. Proper modeling can reduce the order of the simulation dramatically, but higher control accuracy generally tends to increase the order. Hence, a new host of problems have been created, namely, problems associated with the dynamics and control of very high-order systems. The problems can be divided into two intimately related broad classes, the first involving modeling and the second computational aspects.

The modeling problems arise from the fact that for complicated structures the dynamic characteristics, such as spacecraft natural frequencies and modes, cannot be determined with the same accuracy as for simple structures, such as simple uniform beams or membranes. The reason for this is that simple structures generally admit closed-form solutions of the eigenvalue problem. On the other hand, in the case of complicated structures no closed-form solutions exist and one must resort to approximate techniques to compute the natural frequencies and modes, which implies truncation automatically. It is well known that truncation has a profound effect on the computed natural modes, particularly on the higher ones, so that one must distinguish between computed and actual natural frequencies and modes. Hence, whereas in the case of a simple uniform beam truncation does not affect the retained modes, in the case of a complicated structure it does, so that a mathematical model in the form of a simple beam avoids some of the important problems inherent in truncation and cannot be regarded as a realistic model of a complicated structure such as one representing a large flexible spacecraft. Because computed lower modes tend to approach the actual ones as the order of the mathematical model is increased, it follows that a good simulation must be of sufficiently high order to permit accurate determination of the spacecraft modes to be controlled.

The second class of problems arises because of computational difficulties inhererent in the application of various control algorithms to high-order systems. Computational difficulties can be encountered also in the determination of the spacecraft modes, but proper formulation can reduce these to a minimum. Whereas there exists a large variety of algorithms for the control of dynamical systems, these algorithms have been used mostly to control low-order systems, i.e., of order <10. There are reasons to believe that serious computational difficulties are likely to be encountered in the control of high-order systems, both in terms of computational accuracy and computer time. Hence, there appears to be a need to evaluate critically the performance of well-established methods in controlling higher-order systems, as well as to develop new approaches. This paper presents a control scheme capable of controlling a given number of spacecraft modes while recognizing that in reality the order of the system is much higher. The method is based on the concept of modal synthesis advanced earlier by these authors. The number of modes controlled is limited only by the capability of computational algorithms for the solution of the eigenvalue problem for real symmetric matrices, which is ample.

There appear to be several control schemes based on the concept of the system natural modes. Whereas at first sight one may be tempted to think that they are all really the same, a closer examination will reveal significant differences. For example, in discussing general dynamical systems, Brogan[1] refers to *modal decomposition* as the procedure of determining a linear transformation reducing the set of simultaneous ordinary differential equations describing the system behavior to an independent (or nearly independent) set. The procedure amounts simply to finding the Jordan form for the system, which, in turn, implies the determination of the system modal matrix. The implementation of this approach for arbitrary systems has been rare, because of computational difficulties inherent in actually producing the Jordan form. A different approach to the control of multivariable systems, proposed by Simon and Mitter,[2] has come to be known as *modal control* and is defined as control which changes the system eigenvalues to achieve desired control objectives. The algorithm reduces to the inversion of an $l \times l$ matrix, where l is the number of eigenvalues to be shifted. Reference 2 also presents a procedure whereby a smaller group of eigenvalues is shifted at one time. Reference 2 is concerned with arbitrary systems and, although it expresses the unshifted eigenvalues in a Jordan form, it does not seem to pay much attention to decoupling of the closed-loop system. The same philosophy of modal control has been

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proposed by Porter and Crossley,[3] but apparently their work was independent of that of Simon and Mitter and the computational details are different. For example, to determine the control gains, one must solve a determinantal equation leading to a set of nonlinear algebraic equations in the gains, and the solution of the equation is not unique. To avoid computational difficulties inherent in the solution of nonlinear algebraic equations, they present a method referred to as dyadic control, which is applicable to a more restricted class of problems. Once again, the procedure does not imply decoupling. Both the procedure of Ref. 2 and that of Ref. 3 are generally referred to as "poles allocation." Their feasibility for very high-order systems has yet to be demonstrated. Moreover, Refs. 2 and 3 work with well-defined discrete models and are not concerned with truncation problems. A survey of various control techniques and how they might be applied to the control of nonrigid spacecraft is presented by Poelaert.[4] Although Ref. 4 recommends a mathematical model based on the spacecraft modes, the control schemes discussed do not take particular advantage of decoupling possibilities. Moreover, the example considered is of very low order (=4), so that it provides no test for the computational difficulties mentioned earlier. The approach of Ref. 4 is in line with that of Refs. 2 and 3. An attempt to address the problem of control of high-order gyroscopic systems, such as those associated with flexible spacecraft with spinning parts, has been made by Meirovitch et al.[5] The approach of Ref. 5 will be referred to as *modal-space control* and consists of a scheme for individual control of the spacecraft modes, based on a method for uncoupling the system equations of motion by means of the modal matrix in conjunction with a diagonal gain matrix for the controlled modes. Note that the determination of the modal matrix for gyroscopic systems is considerably simpler than that for arbitrary systems. The problem of controlling a limited number of modes of a flexible system is examined by Balas,[6] who raises the question of control and observation "spillover" due to residual (uncontrolled) modes. Reference 6 points out that observation spillover can lead to instability of the closed-loop system. As in Ref. 4, Ref. 6 uses the modes of the flexible distributed system to construct a discrete (in space) model, but makes no attempt to uncouple the discrete equations. Moreover, in using a uniform beam simply supported at both ends as an example, problems arising from truncation effects on the dynamic characteristics of the reduced system are circumvented. The control scheme of Ref. 6 is based on the pole-allocation method of Ref. 2. An essentially similar system is investigated by Rodriguez, Marsh, and Gunter.[7] Once again, no attempt to uncouple the closed-loop system is made and the system (a square uniform membrane) modes are well known and not affected by truncation. Expanding on the approach of Ref. 5, Meirovitch and Öz,[8] and Meirovitch, VanLandingham, and Öz[9] examine the problem of controlling a gyroscopic system via an observer and develop the relation between actual physical controls on the distributed-parameter system and the generalized controls of the discretized (in space) system, respectively. A scheme combining model reduction based upon control performance sensitivity and model error compensation has been proposed by Skelton.[10]

This paper represents a further development of the approach of Refs. 5, 8, and 9 and it addresses the question of spillover effects of the residual modes on the controlled modes as well as the question of control implementation. The method is designed especially to cope with the problem of high dimensionality of the system dynamics. It considers a mathematical model of order $2n$, corresponding to n controlled and n residual modes, where n can reach into hundreds or even thousands. The higher order of the mathematical model is only for the purpose of computing the system natural frequencies and modes on the ground and not for on-board control. This, of course, permits more accurate determination of the controlled modes, as pointed out in the paper. Hence, although truncation effects and model uncertainties are not eliminated completely they are minimized. The control of n modes is a genuine modal control, as the associated modal gain matrix for the controlled modes is diagonal. The proposed control scheme exhibits no observation spillover due to the modeled uncontrolled modes.

II. Hybrid Equations of Motion

Let us consider a flexible spinning body consisting of a central rigid body, a spinning rigid rotor, and a number of flexible appendages. For simplicity, we focus our attention on a single flexible appendage, with summation implied over all such appendages. We shall assume that the central body can oscillate about the point 0 in the rigid body and that the motion of 0 has no bearing on the angular motion. The flexible appendages are assumed to be rigidly attached to the central body. In equilibrium, the central body and the appendages are at rest and the rotor is spinning with the constant angular velocity Ω. The various axes and other necessary quantities are shown in Fig. 1. The following notation has been used:

D_C	= central rigid domain
D_E	= elastic domain
D_R	= domain of rigid rotor
xyz	= set of axes embedded in D_C
$x_Ey_Ez_E$	= set of axes embedded in D_E
$x_Ry_Rz_R$	= set of axes embedded in D_R
r_C	= position vector of a point in D_C
r_E	= position vector of a point in D_E relative to $x_Ey_Ez_E$
r_R	= position vector of a point in D_R relative to $x_Ry_Rz_R$
u	= elastic displacement vector of a point in D_E relative to $x_Ey_Ez_E$
Ω	= steady rotation of axes $x_Ry_Rz_R$ when in equilibrium state
θ_i	= angular perturbations of xyz from inertial space
$\dot{w}_C$	= absolute velocity vector of a point in D_C
$\dot{w}_E$	= absolute velocity vector of a point in D_E
$\dot{w}_R$	= absolute velocity vector of a point in D_R
$\mathbf{\Omega}_C$	= angular velocity vector of xyz
$\mathbf{\Omega}_E$	= angular velocity vector of $x_Ey_Ez_E$
$\mathbf{\Omega}_R$	= angular velocity vector of $x_Ry_Rz_R$

From kinematics, we can write

$$\dot{w}_C = -\tilde{r}_C\Omega_C \tag{1a}$$

$$\dot{w}_E = \dot{w}_{CE} - (\tilde{r}_E + \tilde{u})\Omega_E + \dot{u} \tag{1b}$$

$$\dot{w}_R = \dot{w}_{CR} - \tilde{r}_R\Omega_R \tag{1c}$$

where $\dot{w}_{CE}$ is equal to $\dot{w}_C$ evaluated at the point E, etc., and

$$\tilde{r} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \tag{2}$$

Note that the velocity of point 0 was taken as zero. Because D_E is rigidly attached to D_R,

$$\Omega_E = \Omega_C = \Omega_C(\theta_i, \dot{\theta}_i) \tag{3a}$$

Moreover,

$$\Omega_R = \Omega_C + \Omega k \tag{3b}$$

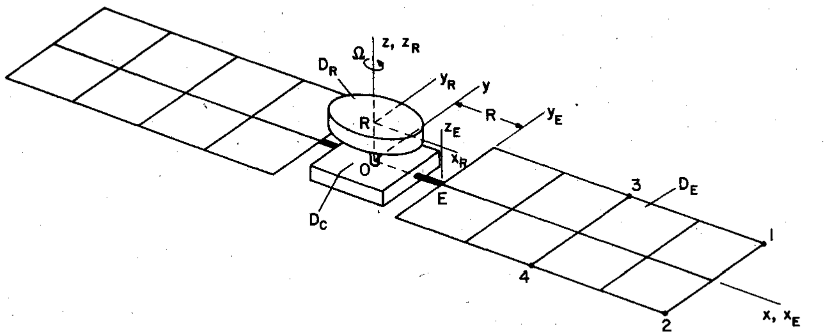

Fig. 1 The flexible gyroscopic system.

where k is a unit vector along z_R, taken to coincide with z. The kinetic energy has the general expression

$$T=\frac{1}{2}\int_{m_C}\dot{w}_C^T\dot{w}_C\mathrm{d}m_C+\frac{1}{2}\int_{m_E}\dot{w}_E^T\dot{w}_E\mathrm{d}m_E+\frac{1}{2}\int_{m_R}\dot{w}_R^T\dot{w}_R\mathrm{d}m_R=T(\theta_i,\dot{\theta}_i,u,\dot{u}) \tag{4}$$

where m_C, m_E, and m_R are the masses of D_C, D_E, and D_R, respectively. The potential energy is assumed to have the symbolic form

$$V=(Au,u)=\frac{1}{2}\int_{D_E}u^TA_{DE}u\mathrm{d}D_E+\frac{1}{2}\int_{S_E}u^TA_{SE}u\mathrm{d}S_E \tag{5}$$

where A is a "two-sided" differential operator defined over D_E and its boundary S_E and containing partial derivatives with respect to the spatial variables. In the absence of elastic restraints, the operator is defined only over D_E. The operator is symmetric and positive definite and can be identified as an *energy operator*.

The system is subjected to external forces and control forces. Denoting by f the actual force density vector acting at every point of the spacecraft, the virtual work can be written in the form

$$\delta W=\int_{D_C}f^T\delta R_C\mathrm{d}D_C+\int_{D_E}f^T\delta R_E\mathrm{d}D_E+\int_{D_R}f^T\delta R_R\mathrm{d}D_R \tag{6}$$

where δR_C, δR_E, and δR_R are virtual displacements associated with points in D_C, D_E, and D_R, respectively. Ignoring second-order terms, these virtual displacements can be written in the form

$$\delta R_C=-\tilde{r}_C\delta\theta \tag{7a}$$

$$\delta R_E=-(\tilde{r}_{CE}+\tilde{r}_E)\delta\theta+\delta u \tag{7b}$$

$$\delta R_R=-(\tilde{r}_{CR}+\tilde{r}_R)\delta\theta \tag{7c}$$

where the meaning of $\delta\theta$ and δu is obvious. Introducing Eqs. (7) into Eq. (6), we obtain

$$\delta W=\mathbf{\Theta}^T\delta\theta+\int_{D_E}f^T\delta u\mathrm{d}D_E \tag{8}$$

where

$$\mathbf{\Theta}=\int_{D_C}\tilde{r}_Cf\mathrm{d}D_C+\int_{D_E}(\tilde{r}_{CE}+\tilde{r}_E)f\mathrm{d}D_E+\int_{D_R}(\tilde{r}_{CR}+\tilde{r}_R)f\mathrm{d}D_R \tag{9}$$

is a torque vector. The Lagrangian can be written in the form

$$L=T-V=\int_{D_C}\hat{L}_{DC}\mathrm{d}D_C+\int_{D_E}\hat{L}_{DE}\mathrm{d}D_E+\int_{S_C}\hat{L}_{SC}\mathrm{d}S_C+\int_{S_E}\hat{L}_{SE}\mathrm{d}S_E+\int_{D_R}\hat{L}_{DR}\mathrm{d}D_R+\int_{S_R}\hat{L}_{SR}\mathrm{d}S_R \tag{10}$$

where $\hat{L}_{DR}$, etc., are Lagrangian densities.

Hamilton's principle has the expression[11]

$$\int_{t_1}^{t_2}(\delta L+\delta W)\mathrm{d}t=0 \qquad \delta\mathbf{\Theta}=\delta u=0 \quad \text{at} \quad t=t_1,t_2 \tag{11}$$

Application of Hamilton's principle yields the equations of motion

$$\frac{\partial L}{\partial\theta}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right)=\mathbf{\Theta} \tag{12a}$$

$$\frac{\partial\hat{L}_{DE}}{\partial u}-\frac{\partial}{\partial t}\left(\frac{\partial\hat{L}_{DE}}{\partial\dot{u}}\right)+\mathcal{L}_E[u]=f \quad \text{over} \quad D_E \tag{12b}$$

where $\mathcal{L}_E$ is a linear differential operator vector. Note that partial derivatives with respect to vectors in Eqs. (12) imply derivatives with respect to vector components arranged in vector form. Equation (12b) is subject to boundary conditions to be satisfied at every point of S_E. We refer to Eqs. (12) as the hybrid differential equations of motion, because Eq. (12a) is an ordinary and Eq. (12b) is a partial differential equation. Of course, the equations must be solved simultaneously. The relation between the operators A and $\mathcal{L}$ is given by

$$\int_{D_E}u^TA_{DE}u\mathrm{d}D_E+\int_{S_E}u^TA_{SE}u\mathrm{d}S_E=\int_{D_E}u^T\mathcal{L}[u]\mathrm{d}D_E \tag{13}$$

III. System Discretization

Sets of hybrid differential equations are difficult to handle and it is customary to discretize the system by such methods as the finite-element method, lumped parameter method, or the assumed modes method.[12] We shall choose the latter approach. Hence, we shall represent the elastic displacement vector as a linear combination of space-dependent admissible functions multiplied by time-dependent generalized coordinates in the form

$$u(P,t)=\Phi(P)\zeta(t) \tag{14}$$

where $\Phi(P)$ is a $3\times s$ matrix of admissible functions, in which P denotes the spatial position, and $\zeta(t)$ is an s-vector of generalized coordinates. The admissible functions are known functions and must be from a complete set, i.e., the linear combination should be capable of approximating the displacement pattern to any degree of accuracy. Assuming that the angular perturbations $\theta_i(t)$ $(i=1,2,3)$ are small, they can be regarded as the components of a rotational vector $\theta(t)$.

Next, let us introduce the vector of generalized displacements for the system

$$q(t)=[\theta^T(t) \,|\, \zeta^T(t)]^T \tag{15}$$

where q is a $(3+s)$-vector known as the configuration vector. In terms of this vector, the kinetic energy can be written in the form

$$T=\tfrac{1}{2}\dot{q}^T m\dot{q}+q^T f\dot{q}+\tfrac{1}{2}q^T k_T q \tag{16}$$

where the elements of the matrices m, f, and k_T are

$$m_{jk}=\frac{\partial^2 T}{\partial\dot{q}_j\partial\dot{q}_k} \quad f_{jk}=\frac{\partial^2 T}{\partial q_j\partial\dot{q}_k} \quad k_{Tjk}=\frac{\partial^2 T}{\partial q_j\partial q_k} \tag{17}$$

Moreover, it can be shown that the system potential energy reduces to

$$V=\tfrac{1}{2}q^T k_V q \tag{18}$$

where

$$k_V=\int_{D_E}\Phi^T A_{DE}\Phi \mathrm{d}D_E+\int_{S_E}\Phi^T A_{SE}\Phi \mathrm{d}S_E \tag{19}$$

In addition, introducing

$$\delta u=\Phi\delta\zeta \tag{20}$$

into the integral in Eq. (8), we obtain

$$\int_{D_E} f^T\delta u \mathrm{d}D_E=\int_{D_E} f^T\Phi\delta\zeta \mathrm{d}D_E=Z^T\delta\zeta \tag{21}$$

where

$$Z=\int_{D_E}\Phi^T f \mathrm{d}D_E \tag{22}$$

is a generalized force vector associated with the elastic generalized coordinates. Introducing the system generalized force vector

$$Q=[\Theta^T \,|\, Z^T]^T \tag{23}$$

and considering Eqs. (15), (21), and (22), Eq. (8) can be rewritten in the compact form

$$\delta W=Q^T\delta q \tag{24}$$

The discretized equations can be displayed in the form of the vector Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}}\right)-\frac{\partial L}{\partial q}=Q \tag{25}$$

which has the explicit form

$$m\ddot{q}+g\dot{q}+kq=Q \tag{26}$$

where

$$g=f^T-f=-g^T \tag{27}$$

is the gyroscopic matrix and

$$k=k_V-k_T \tag{28}$$

is the stiffness matrix, which includes the elastic effects and the centrifugal force effects. The matrix m is positive definite by definition and the matrix k is assumed to be positive definite. Systems with singular stiffness matrices (admitting rigid-body modes) can be treated by the procedure of this paper by removing the source of singularity (rigid-body modes). (See, for example, Ref. 13.)

To solve Eq. (26), it is advisable to transform it into a first-order vector equation. To this end, let us introduce the $2n$-dimensional state vector and the associated force vector.[14]

$$x=[\dot{q}^T \,|\, q^T]^T \tag{29a}$$

$$X=[Q^T \,|\, 0^T]^T \tag{29b}$$

as well as the $2n\times 2n$ matrices

$$M=\left[\begin{array}{c|c} m & 0 \\ \hline 0 & k\end{array}\right]=M^T \tag{30a}$$

$$G=\left[\begin{array}{c|c} g & k \\ \hline -k & 0\end{array}\right]=-G^T \tag{30b}$$

where M is positive definite. Using Eqs. (29) and (30), Eq. (26) can be written in the form of the state equation

$$M\dot{x}(t)+Gx(t)=X(t) \tag{31}$$

IV. Control by Modal Synthesis

The special form of Eq. (31) makes it particularly suitable for control by modal synthesis. The eigenvalue problem associated with Eq. (31) has the form

$$\lambda_r Mx_r+Gx_r=0 \tag{32}$$

where λ_r and x_r are a constant scalar and a constant vector, respectively. It can be shown that the eigenvalues are pure imaginary complex conjugates, $\lambda_r=i\omega_r$, $\bar{\lambda}_r=-i\omega_r$, where ω_r are the spacecraft natural frequencies, and the eigenvectors are also complex conjugates, $x_r=y_r+iz_r$, $\bar{x}_r=y_r-iz_r$ $(r=1,2,\ldots,n)$. The eigenvectors are orthogonal with respect to the matrix M, and in some sense with respect to the matrix G.[14] Indeed, arranging the eigenvectors in the real modal matrix

$$P=[y_1\ z_1\ y_2\ z_2\ldots y_n\ z_n] \tag{33}$$

we can write

$$P^T MP=I \tag{34}$$

where I is the unit matrix of order $2n$. It also follows that

$$-P^T GP=\Lambda=\text{block-diag}\begin{bmatrix}0 & \omega_r\\ -\omega_r & 0\end{bmatrix} \tag{35}$$

where Λ plays the role of the Jordan form for the system. It is not diagonal but block-diagonal, with the order of the blocks being 2. This permits the uncoupling of the system into n independent second-order systems, as shown in Sec. V.

Let us consider the linear transformation,

$$x=\sum_{r=1}^{n}[\xi_r(t)y_r+\eta_r(t)z_r]=Pw \tag{36}$$

where

$$w=[\xi_1\ \ \eta_1\ \ \xi_2\ \ \eta_2\ldots\xi_n\ \ \eta_n]^T \tag{37}$$

is a modal state vector, in which ξ_r and η_r are generalized conjugate modal coordinates. Introducing Eqs. (36) into Eq. (31), multiplying on the left by P^T and considering Eqs. (34) and (35), we obtain

$$\dot{w}=Aw+W \tag{38}$$

where

$$W=P^TX \tag{39}$$

is the associated force vector. Equation (38) represents a set of n pairs of first-order differential equations for the conjugate modal coordinates $\xi_r(t)$ and $\eta_r(t)$. In view of this, W is recognized as the modal control vector.

Assuming that the output vector z has dimension m and is related to the state vector by

$$z=C_0x \tag{40}$$

where in general C_0 has dimension $m\leq 2n$, we conclude that

$$z=C_0Pw=Cw \tag{41a}$$

$$C=C_0P \tag{41b}$$

V. Modal-Space Control Implementation

Considering Eqs. (34) and (39), we can write the discrete control vector in the form

$$X=(P^T)^{-1}W=MPW \tag{42}$$

It will prove convenient to partition the matrix P and the modal control vector W as follows:

$$P=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22}\end{bmatrix} \tag{43a}$$

$$W=[W_C^T \mid W_R^T]^T \tag{43b}$$

where P_{ij} $(i,j=1,2)$ are $n\times n$ matrices and W_C and W_R are n-vectors, so that, recalling Eqs. (29b) and (30a), Eq. (42) yields

$$Q=m(P_{11}W_C+P_{12}W_R) \tag{44a}$$

$$0=k(P_{21}W_C+P_{22}W_R) \tag{44b}$$

Equation (44b) can be regarded as a constraint equation to be satisfied by the modal control vector. Indeed, the equation can be used to write

$$W_R=-P_{22}^{-1}P_{21}W_C \tag{45}$$

It can be shown that the columns of P_{22} are independent so that P_{22}^{-1} is guaranteed to exist. Hence, introducing Eq. (45) into Eq. (44a), the generalized control vector becomes

$$Q=m(P_{11}-P_{12}P_{22}^{-1}P_{21})W_C \tag{46}$$

Equation (46) establishes a unique relation between the generalized control vector Q and the modal control vector W_C. Recalling that A is a block-diagonal matrix, we can introduce the notation

$$A=\begin{bmatrix} A_C & 0 \\ 0 & A_R\end{bmatrix} \tag{47a}$$

$$w=[w_C^T \mid w_R^T]^T \tag{47b}$$

which enables us to rewrite Eq. (38) in the form

$$\dot{w}_C=A_Cw_C+W_C \tag{48a}$$

$$\dot{w}_R=A_Rw_R+W_R \tag{48b}$$

Regarding the modal vector W_C as arbitrary, we can choose it so that the generalized modal vector w_C will be controlled in any prescribed manner. Hence, the vector w_C can be identified as being associated with the controlled modes. Consequently, w_R will be identified as the vector associated with the uncontrolled modes. Due to the nature of the matrix A, we shall take n as an even integer.

The control vector W_C can be a linear function of the state vector w, and by Eq. (45) so will the vector W_R. Such control is sometimes referred to as proportional control. The control vector can also be a nonlinear function of the state vector, such as in the case of relay-type control. Both types of controls are discussed in Ref. 5. Care must be exercised, if the controls are to be independent. As seen in Ref. 5, however, this presents no particular difficulty. The control forces associated with each pair of controlled conjugate generalized coordinates are assumed to depend only on these coordinates. In particular, the force associated with the coordinate $\xi_r(t)$ is taken equal to zero and that associated with $\eta_r(t)$ is taken to depend on $\eta_r(t)$ alone. This guarantees complete decoupling of Eqs. (48).

Let us now concentrate on the proportional control and write the relation between the control vector and state vector in the form

$$W=Kw \tag{49}$$

where K is the modal control gain matrix. It will prove convenient to partition the matrix K as follows:

$$K=\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22}\end{bmatrix} \tag{50}$$

were K_{ij} $(i,j=1,2)$ are $n\times n$ matrices, so that Eq. (49) can be rewritten as

$$W_C=K_{11}w_C+K_{12}w_R \tag{51a}$$

$$W_R=K_{21}w_C+K_{22}w_R \tag{51b}$$

Inserting Eqs. (51) into the constraint equation, Eq. (45), and equating the coefficient matrices of w_C and w_R on both sides of the equation, we obtain

$$K_{21}=-P_{22}^{-1}P_{21}K_{11} \tag{52a}$$

$$K_{22}=-P_{22}^{-1}P_{21}K_{12} \tag{52b}$$

But, for independent controls of the type discussed earlier, the submatrices K_{11} and K_{12} must be chosen as follows:

$$K_{11}=\mathrm{diag}[\tfrac{1}{2}(1^r+(-1)^r)c_{2r}] \tag{53a}$$

$$K_{12}=0 \tag{53b}$$

where c_{2r} $(r=1,2,\ldots,n/2)$ are negative constants. Because the matrices K_{21} and K_{22} are related to K_{11} and K_{12} by Eqs. (52), Eqs. (53) define the control gain matrix K fully. Inserting Eqs. (51) into Eqs. (48) and considering the fact that $K_{12}=K_{22}=0$, we can write the matrix equation

$$\begin{bmatrix}\dot{w}_C \\ \dot{w}_R\end{bmatrix}=\begin{bmatrix} A_C+K_{11} & 0 \\ K_{21} & A_R\end{bmatrix}\begin{bmatrix} w_C \\ w_R\end{bmatrix} \tag{54}$$

which illustrates clearly that the vector w_C can be controlled independently of the vector w_R, so that no control spillover due to modeled uncontrolled modes exists. The behavior of the system is controlled by the eigenvalues of A_C+K_{II} and A_R. Because the first have negative real parts and the second have zero real parts the system is stable. Indeed, as time unfolds, the vector w_C reduces to zero while the vector w_R undergoes bounded oscillation.

Perhaps at this point it is appropriate to underscore the difference between the modal control developed here and that proposed by Simon and Mitter.[2] In the present approach, the system of equations is transformed to the modal space and the control laws are designed in the modal space so as to permit independent control of each individual mode. These generalized control forces are then synthesized to obtain the actual control forces. On the other hand, Ref. 2 is essentially a pole-allocation technique leading directly to a control gain matrix in the original coupled space. It appears that the present approach is appreciably simpler to implement, and this is particularly true for high-order systems, such as those arising in connection with flexible spacecraft.

VI. Decoupled Modal Observer

We shall consider a reduced-order deterministic observer of order p, $p\leq 2n$. Clearly, the full-order observer is merely a special case of the reduced-order observer. The differential equation describing the observer behavior is assumed to have the form

$$\dot{\hat{w}}_0=A_0\hat{w}_0+B_0z+NP^TX \tag{55}$$

where $\hat{w}_0$ is the p-dimensional observer state vector, A_0 is a $p\times p$ matrix, B_0 a $p\times m$ matrix, and N a $p\times 2n$ matrix, in which $p+m=2n$. The observer is designed to estimate a linear combination of the modal coordinates $\xi_r(t)$, $\eta_r(t)$ $(r=1,2,\ldots,n)$, so that

$$\hat{w}_0=Lw \tag{56}$$

where L is a $p\times 2n$ transformation matrix. The system output z is as given by Eq. (41a) and can be written in the more explicit form

$$z=Cw=C_Cw_C+C_Rw_R \tag{57}$$

where the second term on the right side represents the contamination of the measurements by the uncontrolled modes.

Next, let us multiply Eq. (38) on the left by L, consider Eq. (39) and write

$$L\dot{w}=LAw+LP^TX \tag{58}$$

so that, subtracting Eq. (58) from Eq. (55), we obtain

$$\dot{\hat{w}}_0-L\dot{w}$$

$$=A_0(\hat{w}_0-Lw)+(B_0C-LA+A_0L)w+(N-L)P^TX \tag{59}$$

where the term A_0Lw has been added and subtracted for convenience. Choosing B_0 and N so as to satisfy

$$B_0C-LA+A_0L=0 \tag{60a}$$

$$N=L \tag{60b}$$

and introducing the p-dimensional error vector

$$\epsilon(t)=\hat{w}_0(t)-Lw(t) \tag{61}$$

Eq. (59) reduces to

$$\dot{\epsilon}(t)=A_0\epsilon(t) \tag{62}$$

which has the solution

$$\epsilon(t)=e^{A_0t}\epsilon(0) \tag{63}$$

Hence, if the observer matrix A_0 is chosen such that its eigenvalues have negative real parts, then

$$\lim_{t\to\infty}\epsilon(t)=0 \tag{64}$$

from which it follows that

$$\lim_{t\to\infty}\hat{w}_0(t)=Lw(t) \tag{65}$$

so that, for large t, the observer state approaches a certain linear combination of the state variables. The magnitude of the real parts of the eigenvalues can be chosen sufficiently large that convergence is almost instantaneous. Because the system is deterministic, this presents no problem. Note that the imaginary parts of the eigenvalues can be taken equal to zero, so that convergence is asymptotic.

Assuming that the matrix A_0 has been chosen, and henceforth can be regarded as known, it remains to choose the matrix B_0 to determine fully the observer, Eq. (55). Indeed, use of Eq. (60a) yields the matrix L, and at the same time the matrix N, as indicated by Eq. (60b). To solve Eq. (60a), let us assume that A_0 is diagonal

$$A_0=\text{diag}[\alpha_j] \tag{66}$$

Moreover, let us introduce the notation

$$B=B_0C=[B_{jr}] \tag{67}$$

where B_{jr} are 1×2 matrices of the form

$$B_{jr}=[b_{1jr}b_{2jr}]\qquad (j=1,2,\ldots,p;\quad r=1,2,\ldots,n) \tag{68}$$

as well as the notation

$$L=[L_{jr}] \tag{69}$$

where L_{jr} are 1×2 matrices of the form

$$L_{jr}=[l_{1jr}\ \ l_{2jr}]\qquad (j=1,2,\ldots,p;\quad r=1,2,\ldots,n) \tag{70}$$

Then, recalling Eq. (35), it can be verified that Eq. (60a) yields the set of equations

$$\omega_rL^*_{jr}-\alpha_jL_{jr}=B_{jr}\qquad (j=1,2,\ldots,p;\quad r=1,2,\ldots,n) \tag{71}$$

where

$$L^*_{jr}=[-l_{2jr}\ \ l_{1jr}]\qquad (j=1,2,\ldots,p;\quad r=1,2,\ldots,n) \tag{72}$$

The solution of Eq. (71) has the form

$$L_{jr}=-\frac{1}{\alpha_j^2+\omega_r^2}B_{jr}\begin{bmatrix}\alpha_j & \omega_r\\ -\omega_r & \alpha_j\end{bmatrix}\quad\begin{pmatrix}j=1,2,\ldots,p\\ r=1,2,\ldots,n\end{pmatrix} \tag{73}$$

Equations (41b) and (56) can be combined into

$$\begin{bmatrix}z\\ \hline \hat{w}_0\end{bmatrix}=\begin{bmatrix}C\\ \hline L\end{bmatrix}w \tag{74}$$

where the coefficient matrix on the right side is a square matrix of order $2n$ and the vector w plays the role of an estimated state vector. Hence, an estimate of the state vector can be expressed in terms of the measurement vector z and the observer state vector $\hat{w}_0$ in the form

$$\hat{w} = T\begin{bmatrix} z \\ \hline \hat{w}_0 \end{bmatrix} \tag{75}$$

where

$$T = \begin{bmatrix} C \\ \hline L \end{bmatrix}^{-1} \tag{76}$$

Considering the case in which there are n measurements, $m=p=n$, partitioning the matrix T into four $n \times n$ submatrices in the form

$$T = \left[\begin{array}{c|c} T_{11} & T_{12} \\ \hline T_{21} & T_{22} \end{array}\right] \tag{77}$$

and using Eq. (41b), the estimated control state vector can be written as

$$\hat{w}_C = T_{11}z + T_{12}\hat{w}_0 = T_{11}Cw + T_{12}\hat{w}_0 \tag{78}$$

Moreover, introducing the notation

$$K^* = \begin{bmatrix} K_{11} \\ \hline K_{21} \end{bmatrix} \tag{79}$$

the control vector becomes

$$W = K^*\hat{w}_C = K^*T_{11}Cw + K^*T_{12}\hat{w}_0 \tag{80}$$

so that Eq. (38) can be rewritten in the form

$$\dot{w} = (A + K^*T_{11}C)w + K^*T_{12}\hat{w}_0 \tag{81}$$

Similarly, using Eqs. (41b) and (80), the observer equation, Eq. (55), becomes

$$\begin{aligned} \dot{\hat{w}}_0 &= A_0\hat{w}_0 + B_0Cw + LW \\ &= (A_0 + LK^*T_{12})\hat{w}_0 + (B + LK^*T_{11}C)w \end{aligned} \tag{82}$$

Equations (81) and (82) can be conveniently displayed in the matrix form

$$\begin{bmatrix} \dot{w} \\ \hline \dot{\hat{w}}_0 \end{bmatrix} = \left[\begin{array}{c|c} A + K^*T_{11}C & K^*T_{12} \\ \hline B + LK^*T_{11}C & A_0 + LK^*T_{12} \end{array}\right]\begin{bmatrix} w \\ \hline \hat{w}_0 \end{bmatrix} \tag{83}$$

Next, let us reformulate the problem in terms of the error vector ϵ instead of the observer vector $\hat{w}_0$. To this end, we introduce the transformation

$$\begin{bmatrix} w \\ \hline \hat{w}_0 \end{bmatrix} = \left[\begin{array}{c|c} 1 & 0 \\ \hline L & 1 \end{array}\right]\begin{bmatrix} w \\ \hline \epsilon \end{bmatrix} \tag{84}$$

where the dimensions of the two unit matrices and of the null matrix should be obvious, and note that

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline L & 1 \end{array}\right]^{-1} = \left[\begin{array}{c|c} 1 & 0 \\ \hline -L & 1 \end{array}\right] \tag{85}$$

Hence, introducing Eq. (84) into Eq. (83) and premultiplying both sides by Eq. (85), we obtain

$$\begin{bmatrix} \dot{w} \\ \hline \dot{\epsilon} \end{bmatrix} = \left[\begin{array}{c|c} A + K^*(T_{11}C + T_{12}L) & K^*T_{12} \\ \hline -LA + B + A_0L & A_0 \end{array}\right]\begin{bmatrix} w \\ \hline \epsilon \end{bmatrix} \tag{86}$$

Recalling Eqs. (60a) and (67) and recognizing that

$$T_{11}C + T_{12}L = [1 \ \ 0] \tag{87}$$

Eq. (86) can be reduced to

$$\begin{bmatrix} \dot{w} \\ \hline \dot{\epsilon} \end{bmatrix} = \left[\begin{array}{c|c} A + K^*[1 \ \ 0] & K^*T_{12} \\ \hline 0 & A_0 \end{array}\right]\begin{bmatrix} w \\ \hline \epsilon \end{bmatrix} \tag{88}$$

Equation (88) can be written more explicitly in terms of the controlled modes w_C and the residual uncontrolled modes w_R in the form

$$\begin{bmatrix} \dot{w}_C \\ \dot{\epsilon} \\ \hline \dot{w}_R \end{bmatrix} = \left[\begin{array}{cc|c} A_C + K_{11} & K_{11}T_{12} & 0 \\ 0 & A_0 & 0 \\ \hline K_{21} & K_{21}T_{12} & A_R \end{array}\right]\begin{bmatrix} w_C \\ \epsilon \\ \hline w_R \end{bmatrix} \tag{89}$$

The eigenvalues of the coefficient matrix in Eq. (89) are the eigenvalues of $A_C + K_{11}$, A_0 and A_R. Because the eigenvalues of $A_C + K_{11}$ and A_0 have negative real parts and those of A_R have zero real parts, the closed-loop system is stable, so that *no observation spillover instability due to the modeled residual modes exists.*

The preceding result is in direct contrast to that obtained in Ref. 6, in which the system performance is adversely affected by contamination of observations of the residual modes, leading ultimately to instability. To eliminate observation spillover instability, Ref. 6 suggests filtering the sensors outputs. Clearly, the approach of the present paper does not require filtering, as no instability caused by observation spillover is experienced. In practice a control scheme for a distributed parameter system will be subjected to observation spillover due to an infinite number of higher modes. Because there is no practical way of accounting for an infinity of modes, one can mitigate the situation by considering a high-order finite-dimensional mathematical model and eliminating the spillover from a certain number of higher modes included in the mathematical model. The scheme proposed here has the advantage that it safeguards against observation spillover instability due to a number of modes equal to the number of modes that are controlled. The explanation can be found in the fact that the mathematical model used here considers residual uncontrolled modes, without any penalty on the order of the observer. Indeed, the observer order is the same as the number of controlled modes. Even though the system described in Ref. 6 is nongyroscopic, unlike the one discussed in this paper, similar reduction of observation spillover instability should be possible by the procedure presented in this paper.

At this point it is perhaps appropriate to pause and reflect on the manner in which this paper answers some of the questions concerning modeling and control raised earlier. We note that to control n modes, we consider a mathematical model of order $2n$. But, this is merely for the purpose of determining the dynamic characteristics of the system, an off-line computational process performed on the ground and not on-board. The procedure of using a mathematical model of order $2n$ has the added advantage that it permits a more accurate determination of lower modes of uncontrolled vibration, as can be concluded from the inclusion principle for gyroscopic systems.[15] Hence, although not eliminated completely, truncation effects and modeling uncertainties are minimized. The control of the n modes is truly an uncoupled modal-space control, as the associated modal gain matrix for the controlled modes is diagonal.

VII. Sensors and Actuators Locations

Let us assume that the measurement process consists of measurements of absolute velocities v_z of certain points on the elastic appendages and measurements of nutational motions θ_1 and θ_2 of the central body. The relationships between measurements of absolute velocities performed on elastic appendages and generalized elastic velocities are given in detail in Ref. 9, where it is shown that for a sensor located at any point $P(x,y)$ of an elastic panel, we have

$$V_z = v_z - (\dot\theta_1 - \Omega_C\theta_2)y + (\dot\theta_2 + \Omega_C\theta_1)x = \Phi_z^T\dot\zeta \tag{90}$$

in which Φ_z is a row vector in Φ associated with the component u_z in Eq. (14). Moreover, Ω_C is the uniform spin rate of the system of axes xyz (as well as of axes $x_Ey_Ez_E$). Of course, in the case of dual-spin spacecraft with a despun section, $\Omega_C = 0$. Note from Fig. 1 that, in our case, $x = R + x_E$, $y = y_E$.

We shall consider the case in which there are k_s sensors on the elastic appendages performing k_s measurements in the z direction at points $P_j\ (j=1,2,...,k_s)$. From Ref. 9, we can write

$$V = S_V\dot\zeta \tag{91}$$

where

$$V = [V_{z1} \quad V_{z2} \ldots V_{zk_s}]^T \tag{92}$$

is a k_s-dimensional vector in which

$$V_{yi} = V_y(P_i,t) \qquad (i=1,2,...,l_s) \tag{93a}$$

$$V_{zj} = V_z(P_j,t) \qquad (j=1,2,...,k_s) \tag{93b}$$

and

$$S_V = [\Phi_z(P_1) \quad \Phi_z(P_2) \ldots \Phi_z(P_{k_s})]^T \tag{94}$$

is a $k_s \times (n-2)$ matrix.

Alternatively, if elastic displacements relative to the frame $x_Ey_Ez_E$ are measured at the same points, then we can write

$$U = S_U\zeta \tag{95}$$

where

$$U = [U_{z1} \quad U_{z2} \ldots U_{zk_s}]^T \tag{96}$$

in which the meaning of the components is obvious, and S_U has the same form as S_V.

Next, let us define the measurement vector in the form

$$z = [\dot\theta_1 \dot\theta_2 \mid V^T]^T \tag{97}$$

Then, using Eq. (91), the explicit form of the matrix C in Eq. (41a) can be shown to be

$$C = \left[\begin{array}{cc|c|c} 1 & 0 & 0 & 0 \\ 0 & 1 & & \\ \hline \multicolumn{2}{c|}{0} & S_V & 0 \end{array}\right] P \tag{98}$$

From Ref. 9 we can also obtain the relationships between the components of the discretized generalized force vector Q, Eq. (23), and the actual control forces and torques in the form

$$\Theta_1 = \sum_{i=1}^{m_a} M_{xi}(P_i,t) + \sum_{j=1}^{k_a} y(P_j)F_{zj}(P_j,t) \tag{99a}$$

$$\Theta_2 = \sum_{i=1}^{m_a} M_{yi}(P_i,t) - \sum_{j=1}^{k_a} x(P_j)F_{zj}(P_j,t) \tag{99b}$$

$$Z^T = \sum_{j=1}^{k_a} F_{zj}(P_j,t)\Phi_z^T(P_j) \tag{99c}$$

where m_a represents the number of torquers about x and y and k_a denotes the number of thrusters in the z direction. It will prove convenient to introduce the notation

$$M_x = \sum_{i=1}^{m_a} M_{xi}(P_i,t) \qquad M_y = \sum_{i=1}^{m_a} M_{yi}(P_i,t) \tag{100a}$$

$$F = [F_{z1} \quad F_{z2} \ldots F_{zk_a}]^T \tag{100b}$$

$$y_P = [y(P_1) \quad y(P_2) \ldots y(P_{k_a})]^T \qquad x_P = [x(P_1) \quad x(P_2) \ldots x(P_{k_a})]^T \tag{100c}$$

as well as the $(n-2) \times k_a$ matrix

$$E = [\Phi_z(P_1) \quad \Phi_z(P_2) \cdots \Phi_z(P_{k_a})]^T \tag{101}$$

which permits rewriting Eqs. (99) in the compact form

$$\Theta_1 = M_x + y_P^T F \qquad \Theta_2 = M_y - x_P^T F \tag{102a}$$

$$Z = EF \tag{102b}$$

A comparison of Eqs. (94) and (101) reveals that if $k_a = k_s$, and the sensors and actuators are collocated, then

$$E = S_V^T \tag{103}$$

If $k_a \neq k_s$, or the actuators and sensors are not collocated, then E will not be equal to S_V^T, although the two matrices will retain the same structure.

VIII. Derivation of Actual Control Forces and Torques from Discrete Generalized Forces and Modal Control Laws

From Eq. (102b), we can write

$$F = E^\dagger Z \tag{104}$$

where $E^\dagger$ is a pseudo-inverse, which takes into account the possibility that E is not necessarily square. The pseudo-inverse can be chosen in the form[1]

$$E^\dagger = E^T(EE^T)^{-1} \quad \text{if } \dim Z < \dim F \tag{105a}$$

$$E^\dagger = (E^TE)^{-1}E^T \quad \text{if } \dim Z > \dim F \tag{105b}$$

in which Eq. (105a) represents the minimum-norm solution of Eq. (102b) and Eq. (105b) represents the least-squares solution.

It will prove convenient to introduce the notation

$$M = [M_x \ M_y]^T \tag{106}$$

and

$$D = \begin{bmatrix} y_P^T \\ \hline -x_P^T \end{bmatrix} \tag{107}$$

Then, recalling that $\Theta = [\Theta_1 \Theta_2]^T$, we can write Eqs. (102a) in the form

$$\Theta = M + DF \tag{108}$$

so that, considering Eq. (104), the actual torque becomes

$$M = \Theta - DE^\dagger Z \tag{109}$$

Equations (104) and (109) can be combined into

$$\begin{bmatrix} M \\ \hline F \end{bmatrix} = \begin{bmatrix} I & -DE^{\dagger} \\ \hline 0 & E^{\dagger} \end{bmatrix} Q \tag{110}$$

where $Q = [\Theta^T \mid Z^T]^T$ is the generalized control vector.

From the foregoing, we conclude that there is no unique set of actuators capable of producing the same controls. Indeed, there is a choice of number and locations, as long as Eq. (110) is realized. The details of obtaining the vector Q by modal synthesis was discussed in Sec. V.

Equation (104) provides the clue as to the number of actuators on the elastic appendages required. The actual forces are transformed into generalized forces by means of Eq. (102b), so that the equation

$$Z = EF = E\dagger Z \tag{111}$$

must be satisfied, which requires that

$$EE\dagger = I \tag{112}$$

Equation (112) can be satisfied exactly only if dim Z = dim F, in which case $E\dagger = E^{-1}$, or in the case (105a). We shall choose the first alternative which implies that the number of actuators on the elastic appendages coincides with the number of generalized elastic coordinates. Hence, for independent mode control, Eq. (104) reduces to

$$F = E^{-1} Z \tag{113}$$

and Eq. (110) becomes

$$\begin{bmatrix} M \\ \hline F \end{bmatrix} = \begin{bmatrix} I & -DE^{-1} \\ \hline 0 & E^{-1} \end{bmatrix} Q \tag{114}$$

Introducing the notation

$$\begin{bmatrix} I & -DE^{-1} \\ \hline 0 & E^{-1} \end{bmatrix} = H \tag{115}$$

and recalling the generalized force vector $X = [Q^T \mid \mathbf{0}^T]^T$, Eq. (114) yields

$$\begin{bmatrix} M \\ \hline F \end{bmatrix} = [H \mid 0] X \tag{116}$$

Finally, using Eqs. (42) and (80) we obtain the actual controls

$$\begin{bmatrix} M \\ \hline F \end{bmatrix} = [H \mid 0] MPK^* w_C(t) \tag{117}$$

where it is recalled that $w_C(t)$ is the decoupled controlled state vector. An estimate $\hat{w}_C(t)$ of $w_C(t)$ is obtained from the observer, so that the actual controls are

$$\begin{bmatrix} M \\ \hline F \end{bmatrix} = [H \mid 0] MPK^* \hat{w}_C(t) \tag{118}$$

and we observe that all the above matrices are real.

IX. Controllability and Observability

Using Eqs. (102b) and (108), the relation between the generalized force vector Q and the actual torque vector M and force vector F is

$$Q = \begin{bmatrix} \Theta \\ \hline Z \end{bmatrix} = H_0^* \begin{bmatrix} M \\ \hline F \end{bmatrix} \tag{119}$$

where

$$H_0^* = \begin{bmatrix} I & D \\ \hline 0 & E \end{bmatrix} \tag{120}$$

Then, introducing the matrix

$$H^* = P^T \begin{bmatrix} H_0^* \\ \hline 0 \end{bmatrix} \tag{121}$$

the modal system, Eqs. (38) and (39), is said to be completely controllable if and only if the $2n \times 2n^2$ controllability matrix

$$\mathbf{C} = [H^* \mid AH^* \mid A^2 H^* \mid \cdots \mid A^{2n-1} H^*] \tag{122}$$

has rank $2n$. The actual control torques and forces must be located so that this condition is satisfied.

The system is said to be completely observable if and only if the $2n \times 2nm$ observability matrix

$$\mathbf{O} = [C^T \mid A^T C^T \mid (A^2)^T C^T \mid \cdots \mid (A^{2n-1})^T C^T] \tag{123}$$

has rank $2n$, where C is given by Eq. (98). This condition is satisfied if C has independent rows.

X. Numerical Example

A. Spacecraft Control by Modal Synthesis

As an illustration, a dual-spin spacecraft (Fig. 1) was considered. The dynamics of such a configuration is given in Ref. 8. The following parameters were used for the spacecraft and rotor (I's in kg-m^2): $I_x = 250.0$, $I_y = 800.0$, $I_z = 1200.0$, $I_{x_R} = 40.0$, $I_{z_R} = 200.0$, $\Omega = 2\pi$ rad-s^{-1}.

The in-plane motions of the elastic appendages admit a rigid-body mode and are associated with the ignorable coordinate θ_3, which is independent of gyroscopic effects and must be controlled separately, so that it was eliminated from the problem formulation. Two nutation angles, θ_1 and θ_2, and four elastic coordinates, ζ_1, ζ_2, ζ_3, and ζ_4, representing the first out-of-plane bending, first torsional, second out-of-plane bending, and second torsional mode, respectively, were included in the simulation. Hence, the total number of generalized coordinates was six, resulting in a twelfth-order system.

The solution of the eigenvalue problem was obtained by the method of Ref. 13, yielding the following spacecraft natural frequencies (ω's in rad-s^{-1}): $\omega_1 = 1.6694$, $\omega_2 = 4.0279$, $\omega_3 = 5.9125$, $\omega_4 = 8.5252$, $\omega_5 = 15.2095$, $\omega_6 = 19.5835$. Of the twelve spacecraft modes, the first six were regarded as controlled modes and the remaining six as uncontrolled, residual modes. Note that any other combination of six modes could be chosen to be controlled and the remainder be treated as residual modes. The modal matrix P of the spacecraft dynamics is tabulated in Ref. 8. The spacecraft was subjected to initial velocities, with the initial displacements assumed to be zero. The control gain matrix K_{11} was chosen as

$$K_{11} = \text{diag}[0.0 \quad -0.4 \quad 0.0 \quad -0.4 \quad 0.0 \quad -0.4]$$

The spacecraft was controlled by implementing a modal observer in the control loop. The observer was of sixth order, as opposed to a plant of twelfth order. The observer accepted as measurements two nutation rates $\dot{\theta}_1$, $\dot{\theta}_2$ and absolute velocities in the z direction from four sensors, $k_s = 4$, located

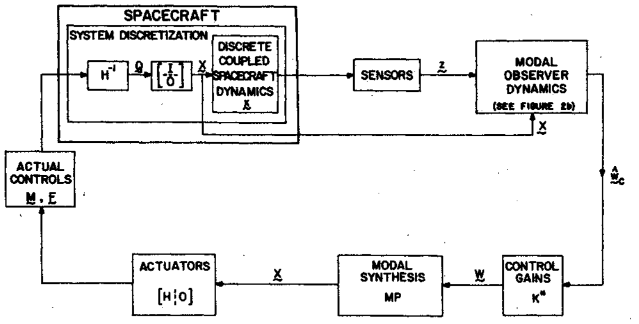

Fig. 2a System control by modal synthesis.

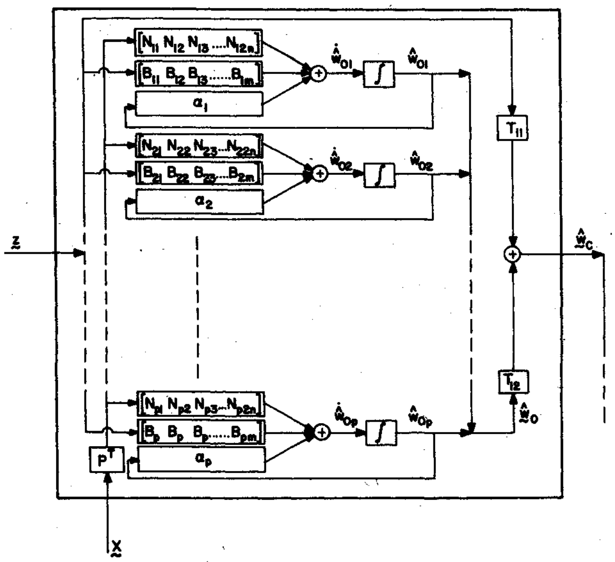

Fig. 2b Decoupled modal observer.

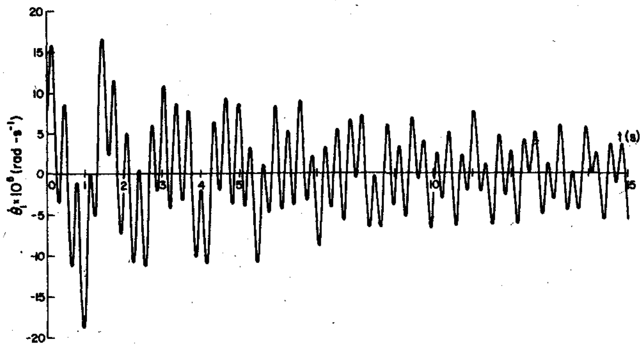

Fig. 3a Nutation rate $\dot{\theta}_I$ time history.

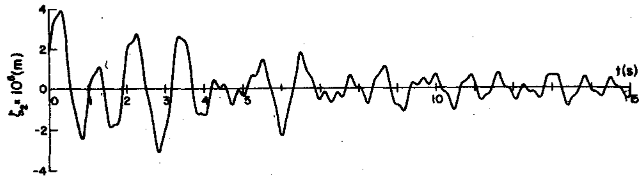

Fig. 3b First torsional mode time history.

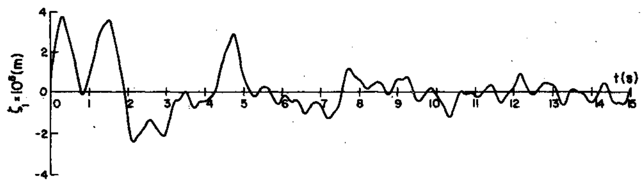

Fig. 3c First out-of-plane bending mode time history.

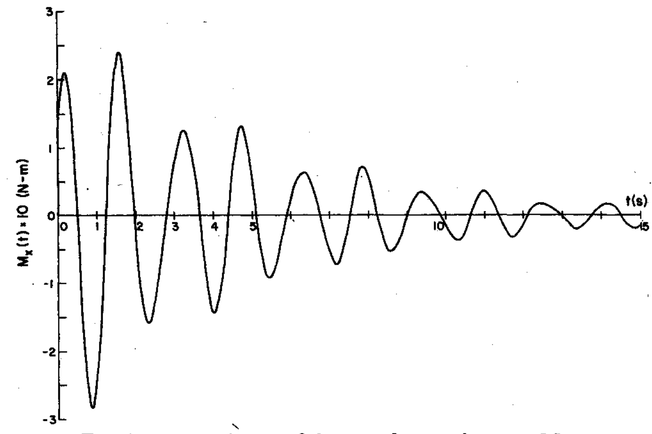

Fig. 4a Time history of the actual control torque M_x.

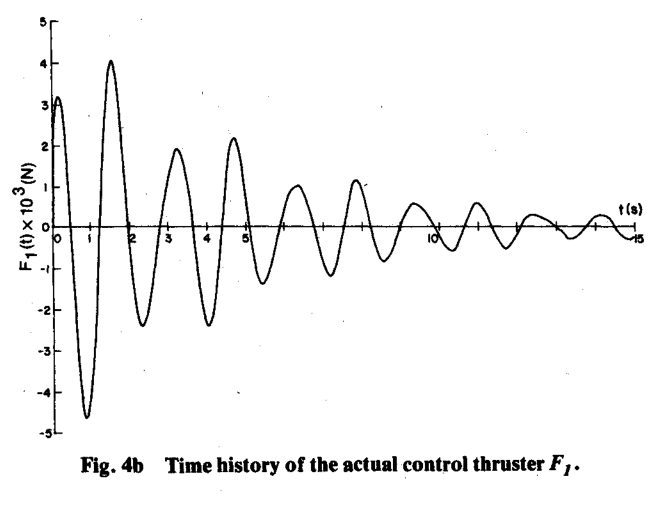

Fig. 4b Time history of the actual control thruster F_I.

on one of the elastic appendages. Figure 1 shows the locations of sensors on the appendage (see points 1, 2, 3, and 4). The measurements were assumed to be contaminated by the residual modes. The observer matrix A_0 and the gain matrix B_0 were taken as follows:

$$A_0 = \mathrm{diag}[-2.0 \quad -3.0 \quad -5.0 \quad -6.0 \quad -7.0 \quad -10.0]$$

$$B_0 = \mathrm{diag}[50.0 \quad 25.0 \quad 0.75 \quad 0.50 \quad 1.0 \quad 1.0]$$

Using the preceding values, the matrix N was calculated according to Eqs. (60b) and (73). Figures 2a-2b show the block-diagrams for the modal control of the system. Figures 3a-3c show the nutation rate $\dot{\theta}_I$, the first torsional mode and the first out-of-plane mode of the appendages, as controlled by the modal observer in the control loop. These time histories show that in this particular example the residual modes do not have any undesirable effect on the system dynamics.

B. Realization of Actual Control Torques and Forces from Independent Modal Control Laws

The modal control input (via modal observer) $W = K^* \hat{w}_C$, as prescribed earlier, was realized by means of four thrusters, $k_a = 4$, taken at the same location as the sensors on the elastic appendage, and a torquer on the central rigid platform, $m_a = 1$, which applied torques about the x and y axes. Although this may not be the case in practice, for simplicity of the formulation it is assumed that the sensors and actuators are collocated. Note that the choice of four thrusters is consistent with the fact that the number of elastic coordinates is four. The control time history for these actuators were obtained by using Eq. (118).

$$[M^T \mid F^T]^T = [M_x(t) M_y(t) F_1(t) F_2(t) F_3(t) F_4(t)]^T = [H \mid 0] MPK^* \hat{w}_C(t)$$

The vector $\hat{w}_C(t)$ was obtained from the observer outputs. The matrix K^* is the control gain matrix given by Eq. (79), which is real and precomputed by using K_{11} given earlier in conjunction with K_{21} according to Eq. (52a). The matrix H, Eq. (115), is also known, given the locations of thrusters. Hence, the prescribed independent modal control laws are physically realizable. The actual control torques and forces were computed as follows:

$$[M^T \mid F^T]^T = 10^3 \times \begin{bmatrix} 0.0 & -0.2555 & 0.0 & 370.8073 & 0.0 & -108.1430 \\ 0.0 & -56.5265 & 0.0 & 0.4565 & 0.0 & -0.1614 \\ 0.0 & 0.0355 & 0.0 & 0.5919 & 0.0 & -0.1974 \\ 0.0 & 0.0363 & 0.0 & -0.5923 & 0.0 & 0.1975 \\ 0.0 & -0.0784 & 0.0 & -1.2055 & 0.0 & 0.3778 \\ 0.0 & -0.0800 & 0.0 & 1.2070 & 0.0 & -0.3778 \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 \\ \hat{\eta}_1 \\ \hat{\xi}_2 \\ \hat{\eta}_2 \\ \hat{\xi}_3 \\ \hat{\eta}_3 \end{bmatrix}$$

We note that the columns of zeros are consistent with the fact that the modal control laws were specified in terms of η-coordinates alone. These zero columns corresponding to ξ-coordinates can be eliminated from the preceding transformation matrix, with the result

$$[M^T \mid F^T]^T = 10^3 \times \begin{bmatrix} -0.2555 & 370.8073 & -108.1430 \\ -56.5265 & 0.4565 & -0.1614 \\ 0.0355 & 0.5919 & -0.1974 \\ 0.0363 & -0.5923 & 0.1975 \\ -0.0784 & -1.2055 & 0.3778 \\ 0.0800 & 1.2070 & -0.3778 \end{bmatrix} \begin{bmatrix} \hat{\eta}_1 \\ \hat{\eta}_2 \\ \hat{\eta}_3 \end{bmatrix}$$

Figures 4a and 4b show the control time histories of the x-component of the torquer, $M_x(t)$, and the thruster in position 1 on the elastic appendage.

XI. Conclusions

A control scheme for distributed gyroscopic systems based on modal-space control is presented. A reduced-order deterministic modal observer is used in the control loop. Actual control forces and torques and sensors positions are specified and the control time histories are obtained from decoupled modal control laws. The dynamical formulation includes controlled and uncontrolled modes simultaneously, resulting in a control scheme free of observation spillover instabilities due to modeled uncontrolled modes. The formulation permits more accurate determination of the controlled modes by recognizing that the physical system is of higher order. Thus, model uncertainties and truncation effects are minimized although not completely eliminated, as a complete elimination of these effects would require an infinite number of degrees of freedom. It should be pointed out that the modal-space control presented here permits implementation not only of linear proportional control but also nonlinear control, such as relay-type on-off control.[5]

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